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FLOW FROM A CHANNEL WITH A FLEXIBLE BARRIER

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The flow from a channel whose outlet is restricted by a flexible inflatable casing is considered. The problem is solved by mapping the flow region onto the half-disk of ancilliary variable variation with allowance for the equation of equilibrium along the flexible contour. A nonlinear integral equation of the Hammerstein type is obtained for the determination of flow. The region of existence and uniqueness of that equation, derived by the successive approximation method, is determined. The obtained solution makes possible the determination of the inflatable casing shape and of pressure near the outflow gap.

Flexible elastic skirts fixed along the perimeter of vehicles floating on air cushions are widely used at present. Such skirts make it possible to maintain a large volume of air in the cushion while freely deflecting over obstacles encountered in the path of their motion. The inflatable skirt attached to the vehicle bottom and connected to the air cushion by an air duct belongs to the type of elastic barriers considered here.

We consider the plane steady flow of a perfect incompressible fluid from an infinite channel (see Fig. 1) whose side $ABCK$ is a flexible mildly sloping casing containing air at pressure p_1. The casing is attached at point A to the body of the air-cushion vehicle (hovercraft). Since the outflow gap h is small and the casing connected to the air cushion by a duct, pressure in the casing along the greater part of AB is equal to the pressure in the air cushion and the AB part of the casing is a straight line inclined to the horizontal at angle $(-\gamma)$. The flow becomes detached from the casing at the end-point C of segment BC. The curved segment BC of the contour is tangent at point B to the straight line AB.

The flow past a flexible profile with fixed ends was first considered in [1], and the model developed there was used in [2] for determining the shape, and the lift and drag coefficients of a sail at various angles of attack. Below we apply the method proposed in [1, 3] for solving flow problems.

The air cushion height H_0 is considerably greater than the outflow gap h (Fig. 1). Hence it is possible to assume that cross section AA (Fig. 1) is at infinity upstream [of the casing], and to reduce the problem to that of outflow from an infinitely large vessel. One wall of the vessel is horizontal and the other inclined at angle $(-\gamma)$ to the horizontal and curved near the outflow gap owing to the difference of pressures in- and outside of the flexible casing. Velocity of the idealized flow at infinity at cross section AA is assumed to be zero. At point D at infinity downstream of the casing the flow velocity V_λ is uniform over the whole width of the stream.

The flow region in the z-plane (Fig. 1) is conformally mapped onto the region of the ancilliary parametric variable, i. e. the half-disk ζ.

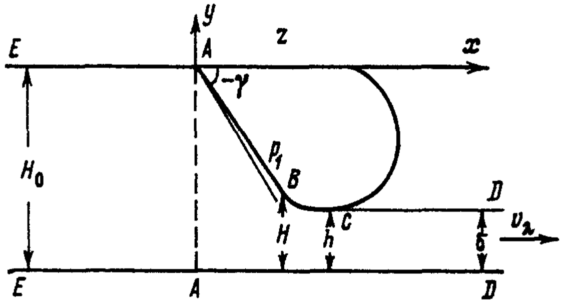

Fig. 1

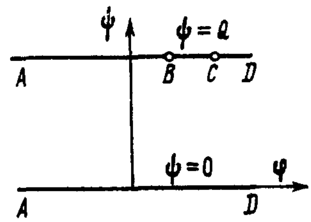

Fig. 2

Points $(i\pi/2, -1, 1)$ at the boundary of the unit half-disk correspond to points A, D, and C of the stream. Specification of these three boundary points determines the uniqueness of the conformal mapping, while variation of the position of point B which corresponds on the circle to parameter $\zeta = e^{i\sigma_0}$ makes it possible to obtain various geometric and kinetic characteristics of the flow shown in Fig. 1. The free surface λ is transformed into the half-disk diameter. Arcs DA and AB of the circle correspond to the straight sections AD and AB and the related arc of the circle corresponds to part BC of the curved flexible contour.

We assume that the streamline $\psi = 0$ corresponds to the supporting surface AD, and streamline $\psi = Q$ corresponds to wall AB, flexible contour BC, and the free surface CD. The region of variation of the complex potential $f = \varphi + i\psi$ has then the form of the infinite band $0 \leqslant \psi \leqslant Q$ (Fig. 2). Specifying the complex potential $f = \varphi + i\psi$ and the auxilliary function

$$\omega = \vartheta + i\tau = \vartheta + i \ln (V / V_\lambda) \tag{1}$$

as functions of the variable ζ provides the complete determination of the flow, since in conformity with (1)

$$df / dz = Ve^{i\vartheta} = V_\lambda e^{-i\omega} = V_\lambda e^{\tau} e^{-i\vartheta} \tag{2}$$

where V is the absolute velocity at point C at coordinate z, ϑ is the angle between the velocity vector and the Ox-axis, and V_λ is the absolute value of velocity at the stream free surface.

The mapping of the band onto the half-disk is defined by formula

$$f = iQ + Q\pi^{-1} \ln [(\zeta^2 + 1)/(\zeta^2 + 2\zeta + 1)] \tag{3}$$

from which in accordance with (2) for the geometric coordinates of the flow we obtain

$$z = \frac{2Q}{\pi V_\lambda} \int e^{i\omega} \frac{(\zeta - 1)}{(\zeta^2 + 1)(\zeta + 1)} \, d\zeta \tag{4}$$

When point ζ lies on the semicircle (Fig. 1), for instance at points that correspond to the flexible contour BC where $\zeta = e^{i\sigma}$, its coordinates are determined by formula

$$z = -\frac{Q}{\pi V_\lambda} \int e^{i\omega} \frac{\operatorname{tg} \sigma \, d\sigma}{1 + \cos \sigma} \tag{5}$$

Within the half-disk ζ function $\omega(\zeta) = \vartheta + i \ln (V / V_\lambda)$ must be regular, because of absence of singularities in the stream, and real at its diameter which corresponds to the stream free surface where $V = V_\lambda$. In region ζ along the circle arc the real part of $\omega(\zeta)$ is continuous everywhere, except at point A, where the direction of velocity is changed by angle $(-\gamma)$ between axis OX and the rectilinear part AB of the taut flexible contour ABC.

We seek function $\omega(\zeta)$ of the form

$$\omega(\zeta) = \omega_0(\zeta) + \Omega(\zeta) \tag{6}$$

where $\Omega(\zeta) = \Phi + iT$ is a function that is analytic in the half-disk, real on its diameter, continuous along the semicircle, with its real part vanishing on the arc DAB. Function $\omega_0(\zeta) = \vartheta_0 + i\tau_0$ has the same properties, except that at point A of the semicircle its real part is discontinuous. Along arc DA (for $\sigma > \pi/2$) $\omega_0(\zeta) = 0$, and along arc ABC (for $\sigma < \pi/2$) $\omega_0(\zeta) = -\gamma$. Along the semicircle function $\omega_0(\zeta)$ has the same properties as function $\omega(\zeta)$. The problem of determination of a function that is analytic in the half-disk, real on its diameter, with its real part having constant discrete values on the semicircle was solved in [3]; for $\zeta = e^{i\sigma}$ on the semicircle we have

$$\begin{gathered}\sigma > \pi/2, \ \vartheta_0 = 0; \ \sigma < \pi/2, \ \vartheta_0 = -\gamma \\ \tau_0 = (\gamma/\pi) \ln |\Lambda(\pi/2, \sigma)| \\ \Lambda(x, y) = \sin[(x-y)/2] / \sin[(x+y)/2]\end{gathered} \tag{7}$$

To determine the second term in formula (6) which defines the unknown function $\omega(\zeta)$ it is necessary to take into consideration the equation of equilibrium of the curved flexible part BC of the casing. The equilibrium equation of that part of the casing which links the pressure drop along the flexible contour and the constant tensioning force T_0 under condition of the casing inextensibility is of the form

$$\Delta p = T_0 \varkappa, \ \varkappa = d\vartheta / ds \tag{8}$$

where $\varkappa$ is the curvature of the flexible contour at a given point and ds is an element of the arc.

Let us determine the pressure drop $\Delta p = p_1 - p$ between the pressure in the casing p_1 and the pressure exercised by the stream at a point of the flexible contour BC. Since the inflatable casing is connected to the air cushion region, pressure p_1 is equal to the pressure at infinity upstream of the casing.

We use the Bernoulli integral $p + {}^1/_2 \, \rho V^2 = p_1$, assuming the velocity at point A to be zero. Since $V = V_\lambda e^\tau$, hence

$$\Delta p = \rho V_\lambda^2 e^{2\tau} / 2$$

From the equilibrium equation (8) we have

$$d\vartheta / ds = \Delta p / T_0 = \rho V_\lambda^2 e^{2\tau} / (2T_0) \tag{9}$$

Since in accordance with (6) $\vartheta = \operatorname{Re} \omega = \vartheta_0 + \Phi$ and for values $\sigma < \pi/2$

which correspond to points of the flexible contour $\vartheta_0 = -\gamma$, along that contour

$$d\vartheta = d\Phi \tag{10}$$

We take the direction along the flexible contour arc from B to C as positive and obtain from (2) for the element ds of the arc the expression

$$ds = |dz| = |df| / (V_\lambda e^\tau) \tag{11}$$

which links the geometric coordinate z of the flow and the complex potential f.

From the condition (9) using (10) and (11) along the flexible contour we obtain

$$d\Phi / |df| = {}^1/_2\, \rho V_\lambda e^\tau / T_0 \tag{12}$$

Along the flexible contour $\zeta = e^{i\sigma}$ with the use of (3) we obtain

$$|df| = -(Q/\pi)\,\mathrm{tg}\,\sigma\,(1 + \cos\sigma)^{-1} d\sigma \tag{13}$$

Since in the motion from point B to point C along the flexible contour $d\sigma <$ 0, hence the last expression is positive.

For the real part of $\Omega = \Phi + iT$ along the flexible contour from (12) and (13) we obtain

$$d\Phi / d\sigma = -(2\pi T_0)^{-1} Q\rho V_\lambda e^\tau\,\mathrm{tg}\,\sigma\,(1 + \cos\sigma)^{-1} \tag{14}$$

For function $\Omega = \Phi + iT$ continuous on the semicircle and real on the diameter the Dini formula, which links the real and imaginary parts (Φ and T, respectively) of function Ω along the circle arc, is valid.

Since it was assumed that for function $\Omega(\zeta)$ with $\sigma > \sigma_0$ $\mathrm{Re}(\Omega) = \Phi(\sigma) = 0$, we can write formula (14) as

$$0 < \sigma < \pi, \quad T(\sigma) = \frac{1}{\pi}\int_0^{\sigma_0} \Phi'(\sigma_1) \ln|\Lambda(\sigma_1, \sigma)|\, d\sigma_1 \tag{15}$$

Using (2) and the expression for τ_0 in (7) we obtain

$$V / V_\lambda = e^{\tau(\sigma_1)} = e^{T(\sigma_1)} \Lambda^{\gamma/\pi} (\pi/2, \sigma_1) \tag{16}$$

Formulas (14) – (16) yield on integral equation of the form

$$T(\sigma) = -\lambda \int_0^{\sigma_0} e^{T(\sigma_1)} \Lambda^{\gamma/\pi}\left(\frac{\pi}{2}, \sigma_1\right) \ln|\Lambda(\sigma_1, \sigma)| \frac{\mathrm{tg}\,\sigma_1}{(1 + \cos\sigma_1)}\, d\sigma_1 \tag{17}$$

$$\lambda = Q\rho V_\lambda / (2\pi^2 T_0), \quad 0 < \sigma < \sigma_0$$

which is satisfied by function $T(\sigma)$ along the flexible contour.

Nonlinear integral equations similar to (17) which contain in the integrand exponents of the unknown function were found in earlier investigations dealing with flow around rigid curvilinear contours [4 – 6]. The method of successive approximations was used for solving problems of this kind.

It should be noted that Eq. (17) is a nonlinear integral equation of the Hammerstein type to which we shall apply the successive approximation process. The canonical form of the Hammerstein equation is

$$f(x) - \int_a^b K(x, y) F(y, f(y))\, dy = 0$$

For the existence and uniqueness of solution of this equation it is necessary that the following three conditions appearing in [7, 8] are satisfied:

$$\varkappa(x) = \left[\int_a^b |K(x, y)|^2\, dy\right]^{1/2} \in L^2(a, b)$$

Function $F(x, v)$ satisfies the Lipschitz condition with respect to the second argument

$$|F(x, v_1) - F(x, v_2)| \leqslant |v_1 - v_2|\, m(x)$$

$$\left(\int_a^b [\varkappa(x)\, m(x)]^2\, dx\right)^{1/2} = c < 1$$

For Eq. (17) functions $f(x)$, $K(x, y)$, $F(y, f(y))$, and $\varkappa(x)$ are of the form

$$f(\sigma) = T(\sigma), \quad K(\sigma, \sigma_1) = -\lambda \ln |\Lambda(\sigma_1, \sigma)|$$

$$F(\sigma_1, T(\sigma_1)) = e^{T(\sigma_1)} \Lambda^{\gamma/\pi}(\pi/2, \sigma_1) \operatorname{tg} \sigma_1 / (1 + \cos \sigma_1)$$

$$\varkappa^2(\sigma) = \lambda^2 \int_0^{\sigma_0} (\ln |\Lambda(\sigma_1, \sigma)|)^2\, d\sigma_1$$

Since in the flow region $V/V_\lambda \leqslant 1$, from (16) we obtain

$$e^{T(\sigma_1)} \Lambda^{\gamma/\pi}(\pi/2, \sigma) \leqslant 1 \tag{18}$$

The fulfilment of the Lipschitz condition by function $F(\sigma, T(\sigma))$ with respect to T means the boundedness of its derivative

$$F_T'(\sigma, T(\sigma)) = e^{T(\sigma)} \Lambda^{\gamma/\pi}(\pi/2, \sigma) \operatorname{tg} \sigma / (1 + \cos \sigma)$$

From (18) follows that

$$F_T'(\sigma, T(\sigma)) \leqslant \operatorname{tg} \sigma / (1 + \cos \sigma) = m(\sigma)$$

which shows that function $F(\sigma, T(\sigma))$ satisfies the Lipschitz condition. The third condition of existence of solution of the Hammerstein type equation as applied to Eq. (17) assumes the form

$$\lambda \left(\int_0^{\sigma_0} \left\{m^2(\sigma) \left[\int_0^{\sigma_0} (\ln |\Lambda(\sigma_1, \sigma)|)^2\, d\sigma_1\right]\right\} d\sigma\right)^{1/2} < 1 \tag{19}$$

This formula makes it possible to estimate for a given σ_0 the applicability region of parameter λ for which the successive approximation process for Eq. (17) is convergent. Thus, for $\sigma_0 = 86°$ $\lambda < 0.147$ and for $\sigma_0 = 88°$ $\lambda < 0.097$.

Successive approximations of the solution of the integral equation (17) are of the form

$$T_0(\sigma) = 0, \quad T_n(\sigma) = -\lambda \int_0^{\sigma_0} \exp(T_{n-1}(\sigma_1))\, \Lambda^{\gamma/\pi}\left(\frac{\pi}{2}, \sigma_1\right) \times \tag{20}$$

$$\ln |\Lambda(\sigma_1, \sigma)|\, d\sigma_1$$

where σ_0 is a constant parameter and $0 < \sigma < \sigma_0$ corresponds to points on the flexible contour.

The calculation method consists of partitioning the interval $(0, \pi)$, within which varies parameter σ, in equal intervals separated by nodes at each of which function $T_n(\sigma)$ is determined by formulas (20). The integrals in (20) are calculated by Simpson's rule. The integrals which define the geometric coordinates of the flow are calculated by the same method.

The coordinates of the flexible contour are determined by formula (5) and are of the form

$$\left(\frac{x}{\delta}, \frac{y}{\delta}\right) = \frac{1}{\pi}\int_{\sigma}^{\sigma_0} (n_1, n_2)\, e^{-T(\sigma)}\Lambda^{\gamma/\pi}\left(\frac{\pi}{2}, -\sigma\right)\frac{\operatorname{tg}\sigma}{1+\cos\sigma}\, d\sigma$$

$$n_1 = \cos(-\gamma + \Phi(\sigma)),\quad n_2 = \sin(-\gamma + \Phi(\sigma))$$

where $\delta = Q / V_\lambda$ is the stream thickness at infinity of the surface. On the semicircle diameter which corresponds to the stream free surface function $\omega(\zeta)$ is real and determined by values of its real part $\vartheta = \vartheta_0 + \Phi$ on the semicircle. By the Schwarz — Poisson formula [4] we have

$$\Phi(x) = \frac{1}{\pi}\int_{0}^{\pi/2} [-\gamma + \Phi(\sigma)]\frac{(1-x^2)}{(1-2x\cos\sigma + x^2)}\, d\sigma \qquad (21)$$

where it has been taken into account that for values of the argument $\sigma > \pi/2$ that correspond to the horizontal supporting surface AD (Fig. 1) $\operatorname{Re}\omega = \vartheta = \vartheta_0 + \Phi(\sigma) = 0$. Along the arc $(0, \sigma_0)$ that corresponds to the flexible contour function $\Phi(\sigma)$ is calculated by integrating formula (14) and, in accordance with the previously indicated condition for $\Omega(\zeta) = \Phi + iT$,

$$\Phi(\sigma) = \pi\lambda\int_{\sigma}^{\sigma_0} e^{T(\sigma_1)}\Lambda^{\gamma/\pi}\left(\frac{\pi}{2}, \sigma_1\right)\frac{\operatorname{tg}\sigma_1}{(1+\cos\sigma_1)}\, d\sigma_1 \qquad (22)$$

The first and sixth approximations of solutions of Eq. (17) are shown in Fig. 3 for $\gamma = 30°$, $\sigma_0 = 86°$, and $\lambda = 0.036$. Note that from the third approximation functions $T_n(\sigma)$ are virtually the same.

The shape of the flexible contour is shown in Fig. 4 for several γ and the same values of σ_0 and λ. Curve *1* in Fig. 4 corresponds to parameters $\gamma = 20°$, $\sigma_0 = 86°$, and $\lambda = 0.036$; values of these parameters for curve *2* in Fig. 4 are the same as in Fig. 3.

Coordinates of points of the free surface obtained from (5) are of the form

$$\left(\frac{x}{\delta}, \frac{y}{\delta}\right) = \left(\frac{x_c}{\delta}, \frac{y_c}{\delta}\right) + \frac{2}{\pi}\int_{0}^{1}(k_1, k_2)\frac{1-x}{(1+x^2)(1+x)}\, dx \qquad (23)$$

$$k_1 = \cos\Phi(x),\quad k_2 = \sin\Phi(x)$$

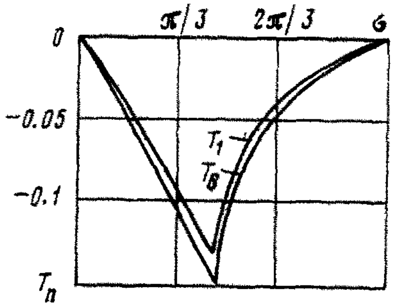

Fig. 3

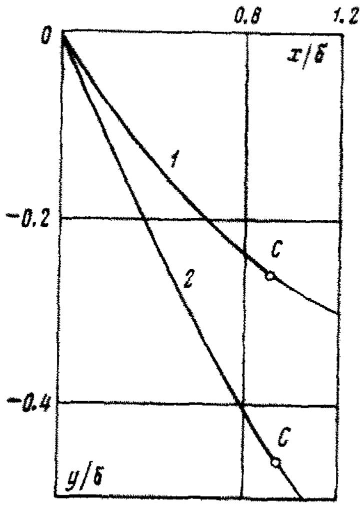

Fig. 4

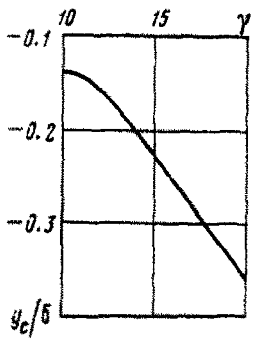

Fig. 5

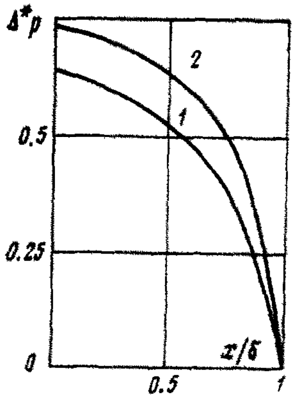

Fig. 6

where x_c and y_c are coordinates of the end point of the flexible profile C at which flow separation occurs, and δ is the stream thickness at infinity downstream of the casing.

As seen in Fig. 4 which shows the shape of the flexible contour near the outflow gap, the curvature of the contour arc is small and it is nearly straight. This is explained by the smallness of parameter λ for which the successive approximation process is convergent. The inversely proportional dependence of λ on the tension force T_0 indicates that the flexible contour is under considerable tension.

Dependence of the dimensionless coordinate y_c / δ of the point of flow separation from the flexible contour on the angle of inclination γ is shown in Fig. 5 for parameters $\lambda = 0.03$ and $\sigma_0 = 88°$.

We determine pressure variation $\Delta^* p = (p - p_a) / (\rho V_\lambda^2 / 2)$ near the outflow gap of the flexible contour using the Bernoulli integral $p + \rho V^2 / 2 = p_a + \rho V_\lambda^2 / 2$, where p_a is the atmospheric pressure, p is the pressure on the flexible contour, and formula (16) which defines the ratio V / V_λ. The dependence shown in Fig. 6 relates to the flow pattern appearing in Fig. 4.

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